〔原著〕

# Some Mathematical Issues in the Relationship between Respiratory Exchange Ratio and Carbon Dioxide Pressure during Rebreathing

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**Abstract :** Respiratory exchange ratio (*RER*) decreases with increasing carbon dioxide pressure ($P$co$_2$) during rebreathing with an equation $y = -K_1x + L_1$, where $x$ and $y$ are $P$co$_2$ and *RER*, respectively, and $K_1$ and $L_1$ are positive constants. The linear regression of *RER* on $P$co$_2$ in a different condition is $y = -K_2x + L_2$. The two regression lines intersect at a point. If the third regression line in another condition passes the crossing point, the regression line can be written as $y = -\{(K_1 + kK_2)/(1 + k)\}x + (L_1 + kL_2)/(1 + k) = -K_3x + L_3$ with a constant k. We found that there are three conditions for k to characterize the slope ($K_3$) and intercept ($L_3$) of the third regression line: (1) $k < -1$, then $K_3 < K_2 < K_1$ and $L_3 < L_2 < L_1$, (2) $-1 < k < 0$, then $K_2 < K_1 < K_3$ and $L_2 < L_1 < L_3$, and (3) $0 < k$, then $K_2 < K_3 < K_1$ and $L_2 < L_3 < L_1$. Physiological meanings of the crossing point of the regression lines and the constant k are also discussed.

**Keywords :** rebreathing, respiratory exchange ratio, mathematical analyses

## Introduction

Rebreathing is a closed respiration system to inspire exhaled gas using a bag. Because of the closed system oxygen is consumed and carbon dioxide is accumulated in a rebreathing bag. When rebreathing is started with a bag containing room-air, $O_2$ concentration in the bag linearly falls and $CO_2$ concentration exponentially rises, which are due to the partial pressure differences between alveolar gas and mixed venous blood (Mochizuki *et al*., 1984; Uchida *et al*., 1986). On the bases of such concentration changes, respiratory exchange ratio (*RER*) during rebreathing was shown to be lineally decreased with increasing $CO_2$ pressure ($P$co$_2$) in the bag (Uchida, 2001). The slope of the linear regression of *RER* on $P$co$_2$ at rest is altered by exercise, because oxygen uptake ($\dot{V}$o$_2$) and carbon dioxide output ($\dot{V}$co$_2$) are stimulated by exercise. The two regression lines at rest and during exercise intersect at a point. We found that the third regression line during exercise with different intensity passes the crossing point under three conditions. In the present paper physiological meanings of the crossing point and the three conditions are also discussed.

## Linear regression of *RER* on $P$co$_2$

A linear fall of $O_2$ concentration ($F$(t)) and an exponential rise of $CO_2$ concentration ($G$(t)) during rebreathing are described as

$$F(t) = A - Bt \quad , \tag{1}$$

$$G(t) = G_\infty - (G_\infty - G_0)\exp(-Ct) \quad , \tag{2}$$

where A, B and C are positive constants and $G_0$ and $G_\infty$ represent $CO_2$ concentrations at the start and the end of rebreathing, respectively. Based on these equations we have a negative linear regression of *RER* on $P$co$_2$ during rebreathing (Uchida 2001)

$$RER = -(C/B)\ P\text{co}_2(t)\ /(\ P_B - 47) + (C/B)G_\infty \quad , \tag{3}$$

where $P$co$_2$(t) is a time-dependent $CO_2$ pressure in a re-

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breathing bag and $P_B$ is a barometric pressure.

Equation (3) can be written as

$$y = -Kx + L \quad , \tag{4}$$

with K and L as positive constants and $x$ and $y$ correspond to $P\text{co}_2$ (t) and *RER*, respectively (Fig. 1).

The intercepts on the $x$ and $y$ axes are L/K and L, respectively. Because *RER* is defined as $\dot{V}\text{co}_2/\dot{V}\text{o}_2$, and increases in $\dot{V}\text{o}_2$ and $\dot{V}\text{co}_2$ by exercise are not the same during rebreathing, the slope K and intercept L at rest ($K_1$ and $L_1$) are different from those during exercise ($K_2$ and $L_2$) :

$$\text{at rest} \qquad y = -K_1x + L_1 \quad , \tag{5}$$

$$\text{during exercise} \qquad y = -K_2x + L_2 \quad . \tag{6}$$

Experimentally we showed that $K_1 > K_2$ and $L_1 > L_2$ (Ito and Uchida, unpublished data).

## Crossing point of the two regression lines

As shown in Fig. 2 coordinates of the crossing point of the two regression lines Eqs. (5) and (6) are ($p$, $q$), where

$$p = (L_1 - L_2)/(K_1 - K_2) \quad , \tag{7}$$

$$q = (K_1L_2 - K_2L_1)/(K_1 - K_2) \quad . \tag{8}$$

In general the equation of a linear line passing through the crossing point of Eqs. (5) and (6) is written as (Akiyama 1991)

$$K_1x + y - L_1 + k(K_2x + y - L_2) = 0 \tag{9}$$

with a constant k. The following three specific conditions for k should be excluded; k = 0, – 1 and $-K_1/K_2$. If k = 0, the third line coincides with that described by Eq. (5). If k = – 1 and $-K_1/K_2$, the third line is parallel to the $y$ and $x$ axes, respectively (Fig. 3).

## Characteristics of the third regression line

Equation (9) is arranged to

$$y = -\{(K_1 + kK_2)/(1 + k)\}x + (L_1 + kL_2)/(1 + k). \tag{10}$$

Since $K_1 > K_2 > 0$ and $L_1 > L_2 > 0$, $K_1$ and $L_1$ are written as

$$K_1 = K_2 + m \quad , \tag{11}$$

$$L_1 = L_2 + n \tag{12}$$

with positive constants m and n.

Eliminating $K_1$ and $L_1$ from Eq. (10) with Eqs. (11) and (12), we have

$$y = -\{K_2 + m/(1 + k)\}x + L_2 + n/(1 + k) = -K_3x + L_3 \quad . \tag{13}$$

Here,

$$K_3 = K_2 + m/(1 + k) \quad , \tag{14}$$

$$L_3 = L_2 + n/(1 + k) \quad . \tag{15}$$

The $K_3$ and $L_3$ are larger than $K_2$ and $L_2$, respectively, if m/(1 + k) and n/(1 + k) are positive, and smaller if m/(1 + k) and n/(1 + k) are negative. Figure 4 shows how m/(1 + k) or n/(1 + k) varies with k. If k > – 1, m/(1 + k) and n/(1 + k) > 0, and therefore $K_3 > K_2$ and $L_3 > L_2$.

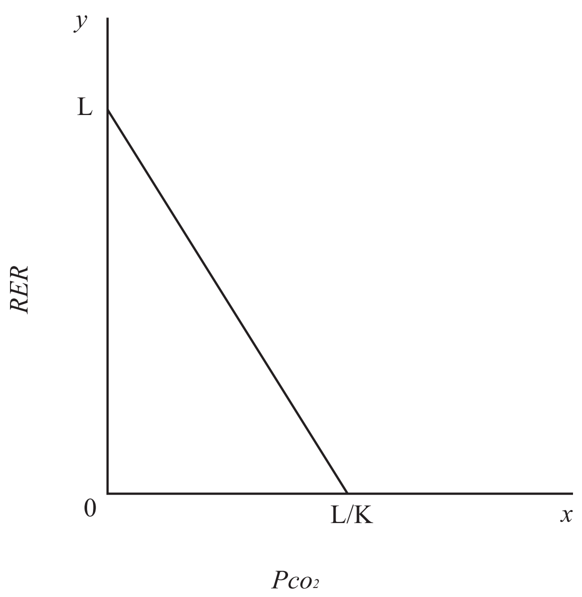


**Fig.1 Linear regression of *RER* on *P*co₂ written as *y* = – K*x* + L, where *x* and *y* correspond to *P*co₂ and *RER* , respectively and K and L are positive constants.**

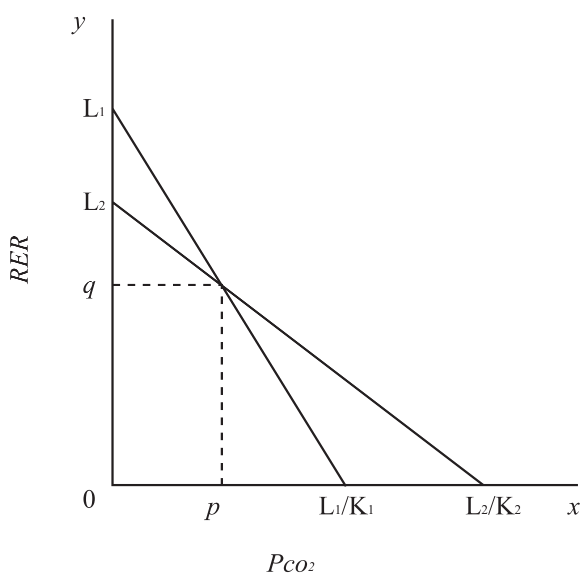


**Fig.2 Linear regression lines in different conditions: $y = -K_1x + L_1$ at rest and $y = -K_2x + L_2$ during exercise. The coordinates of a crossing point of the two regression lines is ($p$, $q$), where $p = (L_1 - L_2)/(K_1 - K_2)$ and $q = (K_1L_2 - K_2L_1)/(K_1 - K_2)$.**

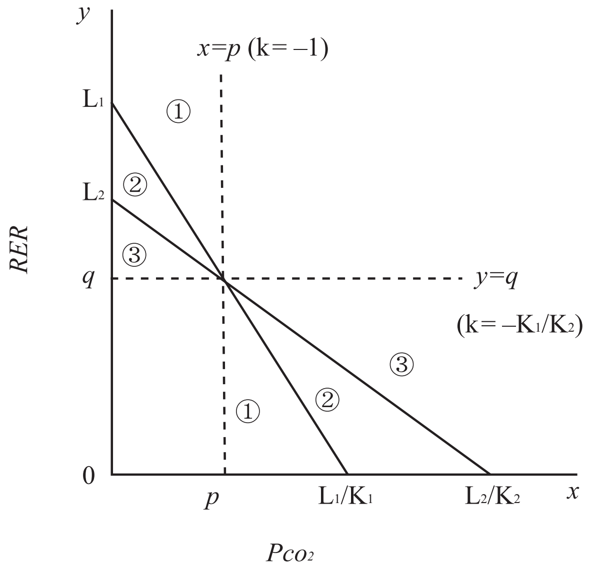


**Fig.3 The third regression line in another condition can be written as $y = -(K_1 + kK_2)/(1 + k)x + (L_1 + kL_2)/(1 + k)$ with a constant k. For the regions ①, ② and ③, see Table 1.**

If $k < -1$, $m/(1 + k)$ and $n/(1 + k) < 0$, and therefore $K_3 < K_2$ and $L_3 < L_2$.

Eliminating $K_2$ and $L_2$ from Eq. (10) with Eqs. (11) and (12), we have

$$y = -\{K_1 - mk/(1 + k)\}x + L_1 - nk/(1 + k)$$
$$= -K_3 x + L_3 \quad . \qquad (16)$$

Here,

$$K_3 = K_1 - mk/(1 + k) \quad , \qquad (17)$$
$$L_3 = L_1 - nk/(1 + k) \quad . \qquad (18)$$

The $K_3$ and $L_3$ are larger than $K_1$ and $L_1$, respectively, if $mk/(1 + k)$ and $nk/(1 + k)$ are negative, and smaller if $mk/(1 + k)$ and $nk/(1 + k)$ are positive. Figure 5 shows how $mk/(1 + k)$ or $nk/(1 + k)$ varies with k. If $-1 < k < 0$, $mk/(1 + k)$ and $nk/(1 + k) < 0$, and therefore $K_3 > L_1$ and $L_3 > L_1$. If $k > 0$ or $k < -1$, $mk/(1 + k)$ and $nk/(1 + k) > 0$, and therefore $K_3 < K_1$ and $L_3 < L_1$.

In summary, there are three conditions for k to characterize $K_3$ and $L_3$ according to Eqs. (14), (15), (17) and (18) with positive constants m and n (Table 1). The regions where the third regression line exists are divided into ①, ② and ③ in Fig. 3.

## Physiological meaning of the crossing point

Equation (4) shows that *RER* is reduced with increasing $P\text{co}_2$ in a rebreathing bag. When $P\text{co}_2$ is equal to alveolar $P\text{co}_2$ ($P_A\text{co}_2$), *RER* is possibly to be 0.8, which

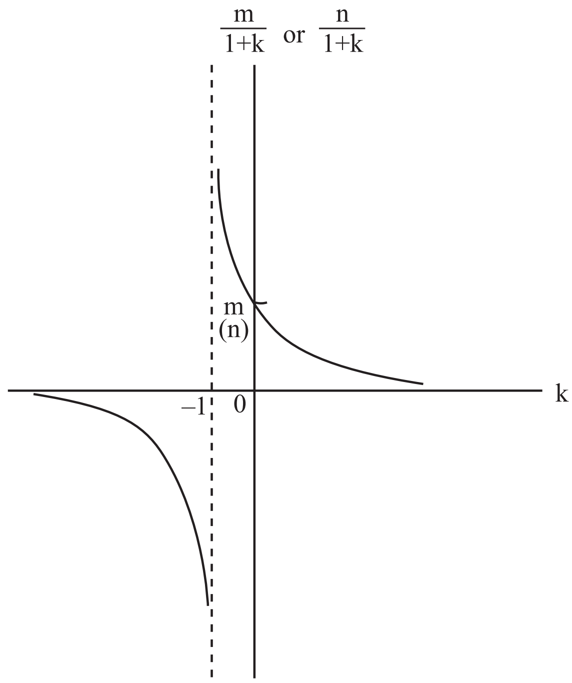


**Fig.4 How $m/(1 + k)$ or $n/(1 + k)$ varies with k.**

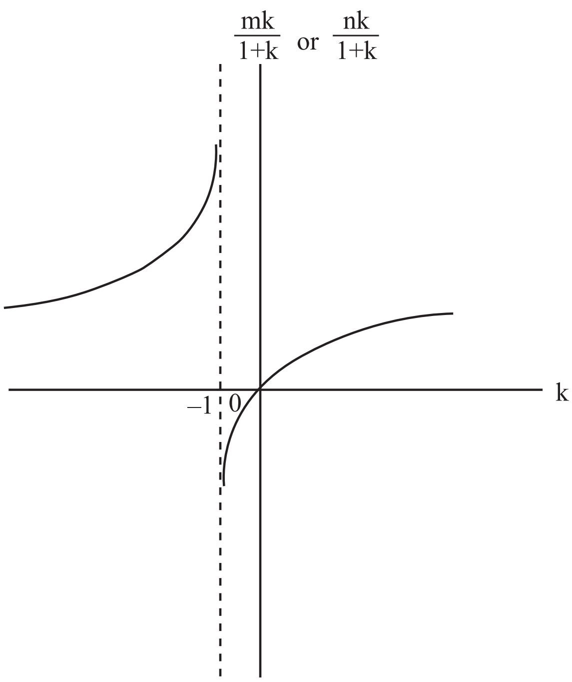


**Fig.5 How $mk/(1 + k)$ or $nk/(1 + k)$ varies with k.**

**Table 1 Characterization of $K_3$ and $L_3$ with the constant k.**

| | $-K_1/K_2 < k < -1$ | $-1 < k < 0$ | $0 < k$ |
|---|---|---|---|
| $m/(1 + k)$ | − | + | + |
| $mk/(1 + k)$ | + | − | + |
| $K_3$ | $K_3 < K_2 < K_1$ | $K_2 < K_1 < K_3$ | $K_2 < K_3 < K_1$ |
| $n/(1 + k)$ | − | + | + |
| $nk/(1 + k)$ | + | − | + |
| $L_3$ | $L_3 < L_2 < L_1$ | $L_2 < L_1 < L_3$ | $L_2 < L_3 < L_1$ |
| | ③ | ① | ② |

The regions in the bottom line exist in Fig. 3.

is a mean respiratory quotient at a resting steady state. Then, from Eq. (5),

$$P_{A}co_2 = (L_1 - 0.8)/K_1 \quad . \tag{19}$$

If *RER* is also 0.8 at $P_{A}co_2$ during exercise,

$$P_{A}co_2 = (L_2 - 0.8)/K_2 \quad . \tag{20}$$

From Eqs. (19) and (20),

$$P_{A}co_2 = (L_1 - 0.8)/K_1 = (L_2 - 0.8)/K_2 \quad . \tag{21}$$

If a/b = c/d = r with constants a, b, c, d and r, we have

$$(a - c)/(b - d) = (rb - rd)/(b - d) = r = a/b = c/d \quad . \tag{22}$$

Therefore, from Eqs. (19), (20) and (21) $P_{A}co_2$ is written as

$$P_{A}co_2 = (L_1 - L_2)/(K_1 - K_2) \quad , \tag{23}$$

indicating that the *x*-coordinate of the crossing point corresponds to $P_{A}co_2$ (Eq. (7)).

**Physiological meaning of the constant k**

According to Table 1, if $-1 < k < 0$, the third regression line exists in the region ① in Fig. 3, if $0 < k$, in the region ②, and if $-K_1/K_2 < k < -1$, in the region ③. Physiologically the slope $K_1$ and the intercept $L_1$ at rest are the largest and the harder exercise intensity, the smaller K and L. Therefore, $K_1$ ($L_1$) at rest is larger than $K_3$ ($L_3$) during exercise, and the third regression line does not exist in the region ①. The constant k reflects the magnitude of exercise intensity and determines the third line exists in whether ② or ③. If $0 < k$, exercise intensity is lower than that in exercise corresponding to Eq. (6), and the third regression line exists in the region ②. If $-K_1/K_2 < k < -1$, exercise intensity is higher than that in exercise corresponding to Eq. (6), and the third regression line exists in the region ③. It can be said that the constant k has a physiological meaning to determine exercise intensity.

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## 要　旨

再呼吸中の呼吸交換比（*RER*）は二酸化炭素分圧（$Pco_2$）の上昇と共に低下して $y = -K_1x + L_1$, のように書くことができる。ここで，$x$ および $y$ は，それぞれ $Pco_2$ および *RER* に相当し，$K_1$ および $L_1$ は正の定数である。異なる条件で再呼吸を行ったときの $Pco_2$ に 対する *RER* の回帰直線は，$y = -K_2x + L_2$ となる。この二つの回帰直線は 1 点で交差する。別のもうひとつの条件で得られた回帰直線がこの交点を通るとき，その式は k を定数として $y = -(K_1 + kK_2)/(1 + k)x + (L_1 + kL_2)/(1 + k) = -K_3x + L_3$ と書くことができる。この第 3 の回帰直線を特性付ける k の条件として以下の 3 つの場合があることがわかった： (1) $k < -1$ ならば $K_3 < K_2 < K_1$ および $L_3 < L_2 < L_1$，(2) $-1 < k < 0$ ならば $K_2 < K_1 < K_3$ および $L_2 < L_1 < L_3$, (3) $0 < k$ ならば $K_2 < K_3 < K_1$ および $L_2 < L_3 < L_1$。この交点および定数 k の生理学的意味についても考察されている。

**キーワード：**再呼吸，呼吸交換比，数学的解析